

The phase transition of Axelrod's model revisited

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Axelrod's model with $F = 2$ cultural features, where each feature can assume k states drawn from a Poisson distribution of parameter q , exhibits a continuous nonequilibrium phase transition in the square lattice. Here we use extensive Monte Carlo simulations and finite size scaling to study the critical behavior of the order parameter ρ , which is the fraction of sites that belong to the largest domain of an absorbing configuration averaged over many runs. We find that it vanishes as $\rho \sim (q_c^0 - q)^\beta$ with $\beta \approx 0.25$ at the critical point $q_c^0 \approx 3.10$ and that the exponent that measures the width of the critical region is $\nu^0 \approx 2.1$. In addition, we find that introduction of long-range links by rewiring the nearest-neighbors links of the square lattice with probability p turns the transition discontinuous, with the critical point q_c^p increasing from 3.1 to 27.17, approximately, as p increases from 0 to 1. The sharpness of the threshold, as measured by the exponent $\nu^p \approx 1$ for $p > 0$, increases with the square root of the number of nodes of the resulting small-world network.

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I. INTRODUCTION

Axelrod's model for the dissemination of culture takes into account two key ingredients of social dynamics [1–3], namely, social influence through which people become more similar when they interact and homophily, which is the tendency of individuals to interact preferentially with similar others [4]. Explicitly, in Axelrod's model the individuals are modeled by agents which are strings of cultural features of length F , where each feature can adopt a certain number k of distinct states. The F cultural features of an agent determine its culture. The N agents are fixed at the nodes of a network (regular lattice or complex network) and the interaction between two connected agents takes place with probability proportional to the number of states they have in common and always results in an increase of the similarity between them.

Whereas social influence is a main feature of the standard two-opinion voter model [5] and homophily has been allowed for in the three-opinion constrained voter model [6, 7], the number of states – two and three, respectively – is not a free parameter in those models. As a result, the two-opinion voter model exhibits only consensus absorbing configurations in regular lattices of arbitrary dimension [5] (see, however, [8] for a report of incomplete ordering on small-world networks). The same conclusion holds for the constrained voter model in an infinite one-dimensional lattice (see, e.g., [9–12]) and since increasing the range of the agents' interactions favors the consensus regime [13, 14], we expect this conclusion to hold for regular lattices of higher dimension as well. In Axelrod's model, however, increase of the number of states k for a fixed string length F leads the social dynamics to freeze in multicultural absorbing configurations even in a one-dimensional lattice [10, 11].

In fact, Axelrod's model exhibits two types of absorbing configurations in the thermodynamic limit $N \rightarrow \infty$: ordered configurations, which are characterized by a few

cultural domains of macroscopic size \mathcal{S} , and disordered configurations, where all domains are microscopic [15–19]. By cultural domain we mean a bounded region of uniform culture. The competition between the disorder of the initial configuration that favors cultural fragmentation and the ordering bias of social influence that favors homogenization results in the nonequilibrium phase transition between those two classes of absorbing states [15]. We note that, similarly to the standard percolation [20], the phase transition occurs in the properties of the absorbing states and so it is static in nature [11].

In this paper we reexamine the nonequilibrium phase transition of a variant of Axelrod's model in which the initial states of the F cultural features of the agents are drawn randomly from a Poisson distribution of parameter $q \in [0, \infty)$,

$$P_k = \exp(-q) \frac{q^k}{k!} \quad (1)$$

with $k = 0, 1, 2, \dots$. In the original model, these states are chosen randomly from a uniform distribution on the integers $1, 2, \dots, q$ [4]. In the square lattice, the Poisson variant exhibits a continuous phase transition for $F = 2$ and a discontinuous one for $F > 2$ [15]. In the one-dimensional lattice, only the disordered regime exists for $F = 2$ and a discontinuous transition between the disordered and the ordered regimes is observed for $F > 2$ [11]. Here we focus on the case $F = 2$ only and study the effect of long range interactions on the standard order parameter of Axelrod's model $\rho = \langle \mathcal{S}_{max} \rangle / N$ that measures the fraction of agents that belong to the largest cultural domain, whose size is denoted by \mathcal{S}_{max} , averaged over many independent runs (see, e.g., [11, 14, 15]).

A previous study of the continuous phase transition for $F = 2$ in the square lattice considered as order parameter the mean density of domains $\mu = \langle \mathcal{N} \rangle / N$, where \mathcal{N} denotes the number of domains of an absorbing configuration, and showed that it vanishes as $\mu \sim (q - q_c^0)^{\beta'}$

with $\beta' = 0.67 \pm 0.01$ at the critical point $q_c^0 = 3.10 \pm 0.02$ [19]. We note that μ is not an order parameter for the standard percolation since it is continuous and non-zero at the threshold [20]. The advantage of considering μ is that it is nonzero in the disordered regime where the convergence of the dynamics to the absorbing configurations is very fast as compared to the convergence to the quasi-consensus configurations of the ordered regime (see, e.g., [12]). The present study complements that analysis by showing that the more usual order parameter ρ vanishes at the critical point as $\rho \sim (q_c^0 - q)^\beta$ with $\beta = 0.25 \pm 0.02$.

In addition, here we study the effect of long range interactions by considering small-world networks in which each link of the square lattice with periodic boundary conditions is rewired with probability p . Rewiring of a link is done by replacing the original neighbor of a given site by a random site chosen uniformly among all possible sites that avoid self-loops and link duplication [21]. The average degree of the resulting network is 4 regardless of the value of p . The extremes $p = 0$ and $p = 1$ correspond to the regular square lattice and to a random network, respectively. Since the underlying regular lattice used to generate the small-world networks is the square lattice we can introduce the length $L \equiv N^{1/2}$, although it has no geometrical meaning for $p > 0$. We find that the transition is discontinuous for $p > 0$ and that the critical exponent that determines the width of the critical region for finite L is $\nu^p = 1.0 \pm 0.05$. However, in the case $p = 0$ where the transition is continuous we find $\nu^0 = 2.1 \pm 0.1$.

The remainder of the paper is organized as follows. For the sake of completeness, in Section II we present a brief account of Axelrod's model [4] and of the Watts and Strogatz algorithm for constructing small-world networks [21]. In Section III we study the behavior of the order parameter ρ near the critical region for three network topologies: the square lattice ($p = 0$), random networks ($p = 1$) and small-world networks with rewiring probability $p = 0.1$. Finally, Section IV offers our concluding remarks.

II. THE MODEL

The Poisson variant of Axelrod's model differs from the original model only by the procedure that generates the cultural states of the agents at the beginning of the simulation: for each feature $l = 1, \dots, F$ of each agent $i = 1, \dots, N$ a state $k = 0, 1, \dots$ is drawn independently using the distribution (1). Once the initial configuration is set, the dynamics proceeds as in the original model [4]. In particular, at each time we pick an agent at random – the target agent – as well as one of its neighbors. These two agents interact with probability equal to their cultural similarity, defined as the fraction of common cultural features they have. An interaction consists of selecting at random one of the distinct features, and making the selected feature of the target agent equal to the corresponding feature of its neighbor. This procedure

is repeated until the system is frozen into an absorbing configuration. According to these rules, at an absorbing configuration any pair of neighbors are either identical or completely different regarding their cultural states.

A feature that sets Axelrod's model apart from most lattice models that exhibit nonequilibrium phase transitions [22] is that all stationary states of the dynamics are absorbing states, i.e., the dynamics always freezes in one of these states. This contrasts with lattice models that exhibit an active state in addition to infinitely many absorbing states [23] and the phase transition occurs between the active state and the usually equivalent absorbing states (see [24] for a simple change in the update rule of Axelrod's model that results in dynamically active metastable states).

The implementation of the Watts and Strogatz algorithm [21] for constructing the small-world networks used in our study begins with a square lattice of linear size L with nearest neighbors interactions and periodic boundary conditions (i.e., a torus). Then for every site $i = 1, \dots, N = L^2$ we rewire the link between i and, say, its left neighbor, with probability p . As mentioned before, rewiring of a link is done by replacing the original neighbor of site i (in this case, the left neighbor) by a random site chosen uniformly among all possible sites that avoid self-loops and link duplication. The procedure is then repeated for, say, the top neighbor of every site of the lattice. Note that for $p = 1$ the resulting network is not a classic random network [25], since the rewiring scheme guarantees that any site will be connected to at least 2 other sites, whereas for classical random graphs any site has probability e^{-4} of being isolated from the other $N - 1$ sites. Otherwise, the resulting networks are very similar to classical random graphs.

III. ANALYSIS OF THE ORDER PARAMETER

For a randomly generated initial configuration of the agents' cultures, we follow the dynamics of Axelrod's model until it reaches an absorbing state – this comprises a single run – and then we calculate the size of the largest cultural domain \mathcal{S}_{max} . Average of this quantity over a large number of independent runs (typically 10^4), which differ by the choice of the initial cultural states of the agents as well as by their update sequence, yields the order parameter $\rho = \langle \mathcal{S}_{max} \rangle / N$ we use to characterize the nature of the absorbing configurations. In the case of random ($p = 1$) and small world ($0 < p < 1$) networks we generate a different network for each run.

Figure 1 offers a bird's eye view of the dependence of the order parameter ρ on the parameter q of the Poisson distribution, which determines the number of different cultural states in the initial population, for the three network topologies we will consider here, viz. the square lattice ($p = 0$), small-world networks with $p = 0.1$ and random networks ($p = 1$). This figure reveals a few interesting features of Axelrod's model. First, the presence

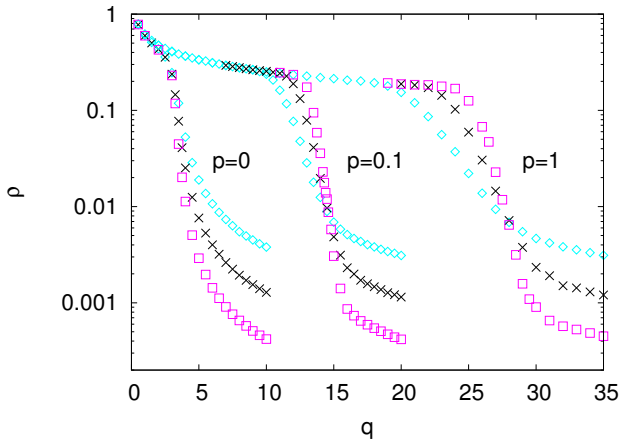


FIG. 1. (Color online) Mean fraction of sites in the largest domain ρ as function of the Poisson parameter q for the square lattice ($p = 0$), small-world networks with $p = 0.1$ and random networks ($p = 1$). The different symbols represent different number of sites $N = L^2$, viz. $L = 100$ (\diamond), $L = 200$ (\times) and $L = 400$ (\square). The error bars are smaller than the symbol sizes.

of long range links favors the ordered regime as indicated by the shift of the threshold region to high values of q , i.e., as the fraction p of long-range links increases, the disorder in the initial configuration must also increase in order the dynamics reaches a disordered absorbing configuration [14]. Second, deep into the ordered phase, say for $q < 10$, the order parameter is completely insensitive to increasing the fraction of long-range links beyond a certain value, say $p = 0.1$. Third, the crossing of the data for different L for $p > 0$ signals the presence of a discontinuous transition in the limit $L \rightarrow \infty$, whereas for $p = 0$ the condition $\rho(L_1) \geq \rho(L_2)$ for $L_1 < L_2$ is satisfied for all values of $q > 0$.

Next we study in detail the behavior of the order parameter ρ in the critical region for the three network topologies exhibited in Fig. 1.

A. Square lattice

The square lattice considered here exhibits only short range interactions and so it offers a baseline for assessing the effects of long-range links on the phase transition of Axelrod's model. In addition, as pointed out before, this study complements previous analyses of the Poisson variant of Axelrod's model in the square lattice [15, 19]. In fact, whereas Ref. [15] considered the qualitative aspects of the continuous nonequilibrium transition (in the sense that there were no attempt to estimate the critical point q_c^0 and the critical exponents), Ref. [19] focused on an alternative order parameter. Since we expect that the location of the critical point is not affected by the choice of the order parameter (in cases there is such choice),

here we borrow from [19] the estimate $q_c^0 = 3.10 \pm 0.02$. The discernment of this resolution will be evaluated by the goodness and consistency of our results.

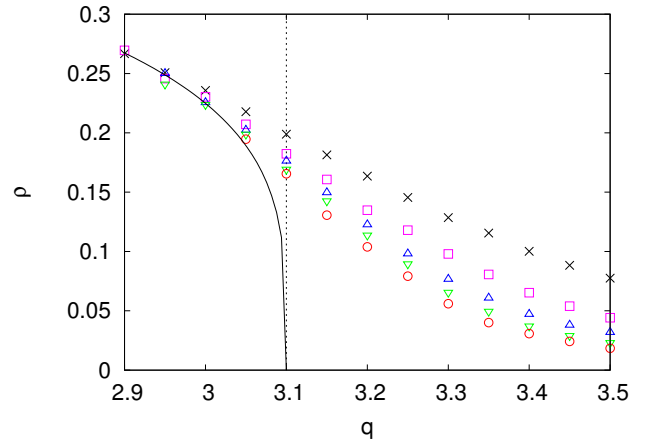


FIG. 2. (Color online) Mean fraction of sites in the largest domain ρ in the critical region for the square lattice ($p = 0$) with linear size $L = 200$ (\times), $L = 400$ (\square), $L = 600$ (\triangle), $L = 800$ (∇), and $L = 1000$ (\circ). The dotted vertical line indicates the location of $q_c^0 \approx 3.1$ and the solid curve is the two-parameters fitting function $\rho = \mathcal{A}(q_c^0 - q)^\beta$ with $\mathcal{A} = 0.40 \pm 0.01$, and $\beta = 0.25 \pm 0.02$. The error bars are smaller than the symbol sizes.

Figure 2 offers a more detailed view of the order parameter ρ near the critical point q_c^0 . Rather remarkably, the finite size scaling theory asserts that for large L the data shown in this figure can be described by the scaling relation [26]

$$\rho \sim L^{-\beta/\nu^0} f_0 \left[L^{1/\nu^0} (q_c^0 - q) \right], \quad (2)$$

where the scaling function is $f_0(x) \propto x^\beta$ for $x \gg 1$ and $\nu^0 > 0$ is a critical exponent that determines the width of the critical region for finite L . Hence in the limit $L \rightarrow \infty$ one has $\rho \sim (q_c^0 - q)^\beta$ near the critical point, where $\beta > 0$ is a critical exponent.

Figure 3 summarizes the results of the fitting of the data for $L = 200$ with the function $\rho = \mathcal{A}(q_c^0 - q)^\beta$, where \mathcal{A} and β are the two adjustable parameters of the fitting. The choice of the fitting region is determined by requiring that the fit curve goes thru the data for $L = 600$ as well. This procedure yields $\beta = 0.25 \pm 0.02$ for the critical exponent that governs the vanishing of the order parameter ρ at the critical point. The resulting fit curve is also shown in Fig. 2 using a linear scale.

According to the scaling relation (2), ρ must decrease to zero as the power law $\rho \sim L^{-\beta/\nu^0}$ at $q = q_c^0$ and we explore this fact in Fig. 4, where ρ is plotted against $1/L$ in a log-log scale, to determine the ratio $\beta/\nu^0 = 0.117 \pm 0.002$. Finally we are now in position to estimate the exponent $\nu^0 = 2.1 \pm 0.1$. The goodness

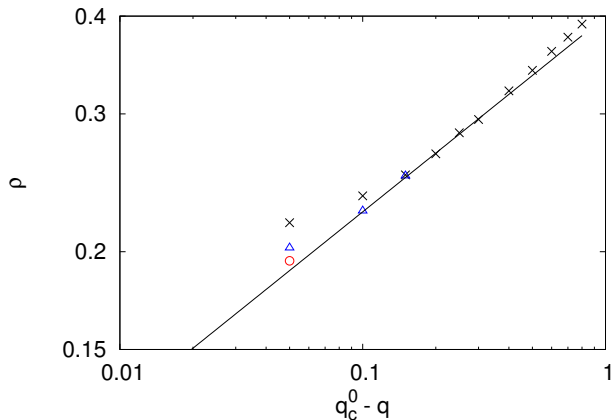


FIG. 3. (Color online) Log-log plot of ρ against $q_c^0 - q$ with $q_c^0 = 3.1$ for the square lattice ($p = 0$) with $L = 200$ (\times), $L = 600$ (\triangle) and $L = 1000$ (\circ). The solid line is the fitting function $\rho = \mathcal{A}(q_c^0 - q)^\beta$ with $\mathcal{A} = 0.40 \pm 0.01$, and $\beta = 0.25 \pm 0.02$.

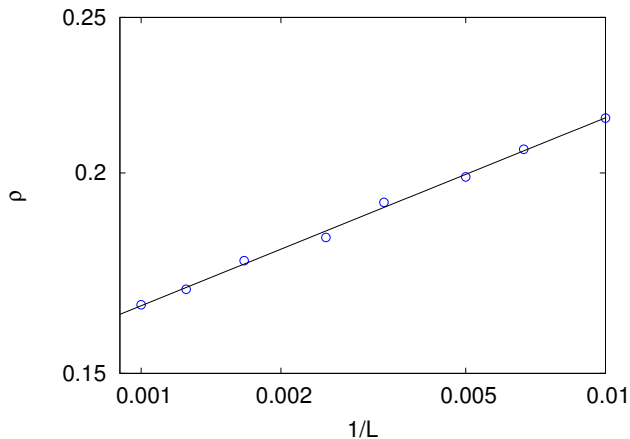


FIG. 4. Log-log plot of ρ against the reciprocal of the linear lattice size at the critical point $q_c^0 \approx 3.1$. The curve fitting the data is $\rho = \mathcal{B}L^{\beta/\nu^0}$ with $\mathcal{B} = 0.371 \pm 0.005$ and $\beta/\nu^0 = 0.117 \pm 0.002$. The error bars are smaller than the symbol sizes.

of our estimates of the critical exponents as well as the judiciousness of borrowing the location of the critical point from [19] can be assessed by checking whether the scaled quantity $L^{\beta/\nu^0} \rho$ is independent of the lattice size L when plotted against the scaled distance to the critical point $L^{1/\nu^0} (q_c^0 - q)$ as predicted by eq. (2). This is shown in Fig. 5. The quality of the collapse of the data for distinct lattice sizes supports heartily our estimates of the critical exponents.

The two critical exponents $\beta \approx 0.25$ and $\nu^0 \approx 2.1$ that determine the behavior of the order parameter ρ in the critical region of Axelrod's model set its continuous

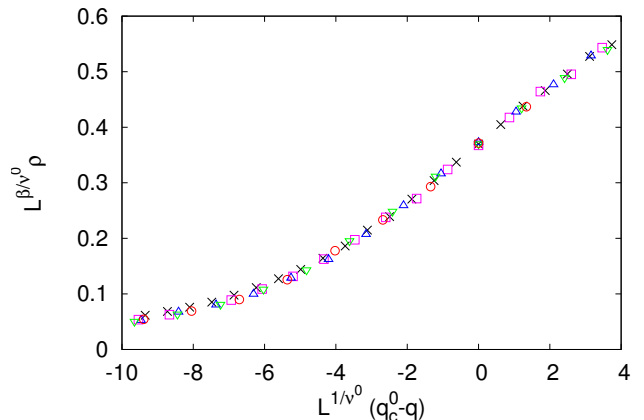


FIG. 5. Scaled order parameter against the scaled distance to the critical point for the square lattice ($p = 0$) with linear size $L = 200$ (\times), $L = 400$ (\square), $L = 600$ (\triangle), $L = 800$ (∇), and $L = 1000$ (\circ). The error bars are smaller than the symbol sizes. The parameters are $q_c^0 = 3.10$, $\beta/\nu^0 = 0.117$ and $\nu^0 = 2.1$.

nonequilibrium phase transition apart from the known universality classes of nonequilibrium lattice models [22]. This is probably due to the static nature of the transition and to the existence of infinitely many absorbing configurations in both – ordered and disordered – phases.

B. Random networks

To study the phase transition for random networks generated by the limit of sure rewiring (i.e., $p = 1$) of the Watts and Strogatz algorithm [21] we need first to obtain a good estimate of the location of the critical point q_c^1 . According to Fig. 1, in the limit $L \rightarrow \infty$ one expects that $\rho \rightarrow 0$ for $q > q_c^1$ and $\rho \rightarrow \rho(q) > 0$ for $q < q_c^1$, so that the order parameter jumps from 0 to ρ_c^1 at $q = q_c^1$. Hence in order to determine q_c^1 , in Fig. 6 we plot ρ against $1/L$ for several values of q in the critical region. A rough estimate of this region is provided by Fig. 1, which also offers a good estimate for the jump at the critical point, $\rho_c^1 \approx 0.18$. At $q = q_c^1$ the order parameter is independent of L (hence the intersection of the data for different L at the critical point), i.e., ρ exhibits a plateau as $1/L \rightarrow 0$. Although such plateaus exist for all $q < q_c^1$ since $\rho \rightarrow \rho(q)$ for large L , there is no risk of confusing the two types of plateaus because the range of ρ shown in Fig. 6 is about one order of magnitude smaller than ρ_c^1 . Visual inspection of Fig. 6 indicates that $27.15 < q_c^1 < 27.2$ so we estimate $q_c^1 = 27.175 \pm 0.125$.

Figure 6 also illustrates the strong finite size effects that hinder the analysis of the order parameter ρ in the critical region. Consider, for instance, the data for $q = 27.5$. If our analysis were restricted to $L \leq 500$ that value of q would be a good candidate for the crit-

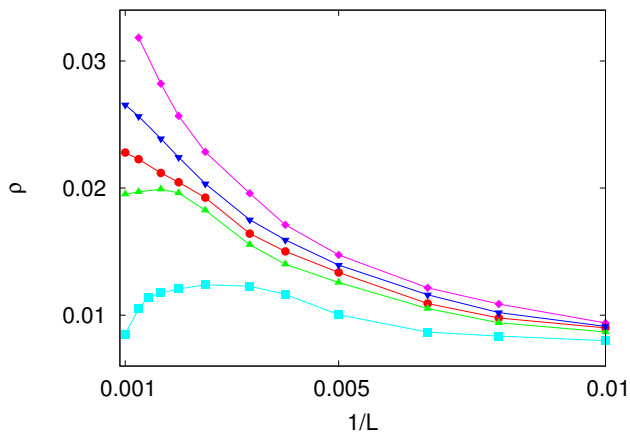


FIG. 6. (Color online) Order parameter ρ against the reciprocal of L for random networks and $q = 27.0$ (\blacklozenge), 27.1 (\blacktriangledown), 27.15 (\bullet), 27.2 (\blacktriangle), and 27.5 (\blacksquare). The critical point is $q_c^1 = 27.175 \pm 0.125$. The error bars are smaller than the symbol sizes and the lines are guides to the eye.

ical point, because of the plateau observed in the range $200 < L < 500$. In fact, the data for $p = 1$ shown in Fig. 1 cross at about $q = 27.5$. Hence to probe the correct critical behavior of ρ we must consider random networks with $L > 600$. This contrasts to our findings for the square lattice, for which the analysis of rather small networks (e.g., $L = 200$) yields useful information about the critical behavior (see Figs. 3 and 5).

We focus now on the characterization of the sharpness of the threshold, i.e., the range of q about q_c^1 where the threshold features persist. To achieve that we will assume that the critical region shrinks to zero like L^{-1/ν^1} as $L \rightarrow \infty$, i.e., we will assume that the order parameter is described by the expression $\rho = f_1 \left[L^{1/\nu^1} (q_c^1 - q) \right]$ in the critical region. Here f_1 is a continuous function such that $f_1(x) \rightarrow 0$ for $x \rightarrow -\infty$ and $f_1(x) \rightarrow \rho_c^1$ for $x \rightarrow \infty$. Our approach is in the same spirit of the finite-size scaling of combinatorial problems [27, 28] (see [29] for a similar study in the context of the quasispecies model), for which there is no geometric criterion for defining a quantity analogous to a correlation length, and so the success of the method in accounting for the size dependence of the order parameter ρ cannot be attributed to the divergence of a correlation length and the consequent onset of a second order phase transition. In addition, we recall that even the parameter L has no geometric interpretation for random networks.

Figure 7 confirms that, in the critical region, the order parameter ρ is a smooth function of the properly rescaled distance to the critical point. The estimate $\nu^1 = 1.0$ for the critical exponent was obtained by requiring that the data for distinct L collapse into a single curve, the function $f_1(x)$. In particular, we find $f_1(0) \approx 0.021 \ll \rho_c^1$, as expected (see Fig. 6). The uncertainty of the estimate

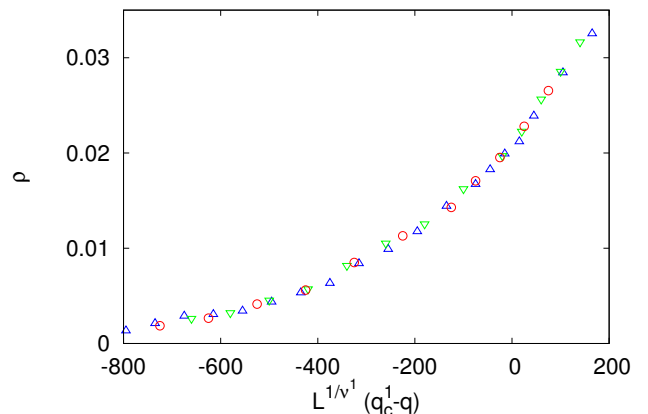


FIG. 7. Order parameter against the scaled distance to the critical point for random networks ($p = 1$) with $L = 600$ (\triangle), $L = 800$ (∇), and $L = 1000$ (\circ). The error bars are smaller than the symbol sizes. The parameters are $q_c^1 = 27.175$, and $\nu^1 = 1.0$.

of ν^1 can be evaluated using the same procedure, i.e., by gauging the quality of the data collapse as the exponent departs from $\nu^1 = 1$. We find very poor data collapses for exponents outside the range $\nu^1 = 1.0 \pm 0.05$ (data not shown). This figure reveals also our difficulty to obtain reliable results in the ordered phase $q < q_c^1$ due to the very long convergence times [12]. We conclude then that the sharpness of the transition increases with the square root of the number of agents, $N^{1/2}$.

C. Small-world networks

We turn now to the analysis of the discontinuous transition for small-world networks with rewiring probability $p = 0.1$. Our aim is to verify whether the exponent ν^p , and hence the sharpness of the threshold, is influenced by the value of p . Figure 1 indicates that in the limit $L \rightarrow \infty$ the order parameter jumps from $\rho_c^{0.1} \approx 0.22$ to 0 at the threshold. The procedure used to determine the critical point is the same as that illustrated in Fig. 6 for random networks and yields $q_c^{0.1} = 14.40 \pm 0.05$. The finite size effects are also very strong in this case and so it is also necessary to resort to large networks, say $L > 500$, to assess the critical region.

In Fig. 8 we examine the assumption that the order parameter satisfies $\rho = f_{0.1} \left[L^{1/\nu^{0.1}} (q_c^{0.1} - q) \right]$ in the critical region. As before, $f_{0.1}$ is such that $f_{0.1}(x) \rightarrow 0$ for $x \rightarrow -\infty$ and $f_{0.1}(x) \rightarrow \rho_c^{0.1}$ for $x \rightarrow \infty$. The best collapse of the data for distinct L is obtained with $\nu^{0.1} = 1.0 \pm 0.05$ and it is shown in the figure. The uncertainty of the exponent is estimated as before, i.e., by varying $\nu^{0.1}$ around 1.0 and gauging the quality of the resulting data collapse. In addition, Fig. 8 shows that

$f_{0.1}(0) \approx 0.015$, implying that the data for different L intersect at $(14.40, 0.015)$ in the plane (q, ρ) .

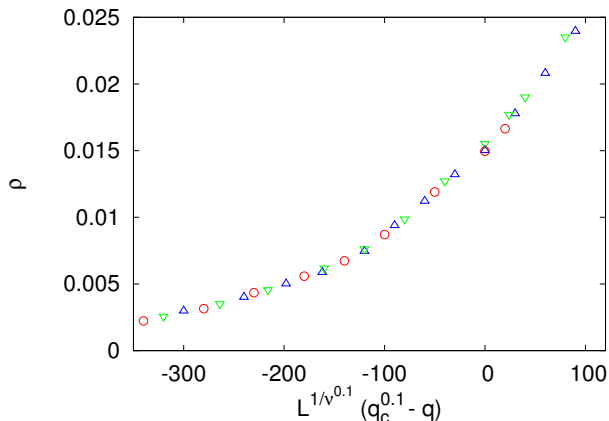


FIG. 8. Order parameter against the scaled distance to the critical point for small-world networks with $p = 0.1$ and $L = 600$ (\triangle), $L = 800$ (∇), and $L = 1000$ (\circ). The error bars are smaller than the symbol sizes. The parameters are $q_c^{0.1} = 14.40$, and $\nu^{0.1} = 1.0$.

We have also considered values of rewiring probability down to $p = 10^{-4}$ and verified that the order parameter ρ when plotted against q for different values of L (see Fig. 1) always cross at some value q_c^p , thus signalling the existence of a discontinuous transition between the ordered ($\rho \geq \rho_c^p$) and the disordered ($\rho = 0$) phases. Although we have estimated the sharpness of the threshold for $p = 1$ and $p = 0.1$ only, the observed irresponsiveness of the exponent ν^p to this tenfold decrease of p prompt us to conjecture that this exponent is a universal feature of the discontinuous transitions of Axelrod's model with two cultural features ($F = 2$) in small-world networks.

IV. CONCLUSION

Axelrod's model exhibits a good balance between simplicity and realism, being considered a choice model to study collective social phenomena from a quantitative perspective [30]. For instance, Axelrod's model has proven useful to study the effects of global media on the polarization of public opinion [31–34] as well as to understand the emergence of a collective intelligence from the local interactions between dumb agents [35, 36]. However, rather than explore the riveting applications of Axelrod's model in social and political science [4], here we focused on the not less enthralling contributions of Axelrod's model to statistical physics [2].

In particular, our aim in this contribution was to offer a quantitative characterization of the critical behavior of the order parameter ρ , which measures the fraction of agents, sites or nodes that belong to the largest cultural

domain of the absorbing configurations. Such quantitative analysis is feasible only for the Poisson variant of Axelrod's model [15], since we need to probe the very close vicinity of the critical point in order to compute the critical exponents that characterize the order parameter. Such detailed study is lacking even for the familiar square lattice [15, 19], as pointed out in Section I.

Here we considered a family of small-world networks for which the square lattice and the random networks are extreme limits of the rewiring probability p . The larger the value of p , the greater the number of long-range links in the network [21]. We found that for $p > 0$ the order parameter ρ exhibits a discontinuity at the critical point q_c^p and that q_c^p increases with increasing p , indicating that the long-range links enlarge the domain of the ordered phase. More pointedly, we found that the transition is discontinuous down to $p = 10^{-4}$ and so we conjecture that it is discontinuous for all $p > 0$. Actually, these qualitative points were already made in Ref. [14], whose authors studied Axelrod's model in several complex networks, small-world networks included, for integer, uniformly distributed initial cultural states and $F = 10$.

The distinctive aspect of our work on small-world networks is that we were able to characterize the critical region, i.e., to estimate the sharpness of the threshold through the critical exponent ν^p (see Section IIIB). Our finding that $\nu^p \approx 1$ for $p = 1$ and $p = 0.1$ implies that the sharpness of the discontinuous transition increases with $N^{1/2}$ where N is the number of agents. We expect that this result holds true for all $p > 0$. In fact, if the value of ν^p were to depend on p , this dependence should be observed with the tenfold decrease of p shown in Figs. 7 and 8.

Our findings regarding the critical behavior of ρ in the square lattice for which the nonequilibrium phase transition is continuous [15, 19] corroborate its unique character: the critical exponents $\beta = 0.25 \pm 0.02$ and $\nu^0 = 2.1 \pm 0.1$ set it apart from the known universality classes of nonequilibrium lattice models [22]. This may be due to the distinctive static character of the transition, which separates two types of absorbing configurations that differ on their distributions of domain sizes.

The manner we produced the discontinuous transitions in Axelrod's model with $F = 2$ cultural traits was through the introduction of long-range links on the basal square lattice. However, there is another, perhaps more natural, way to produce those transitions in the square lattice, namely, by increasing the number of cultural traits beyond $F = 2$ [15]. Although this approach would result in a considerable raise on the computational cost to simulate Axelrod's model, the unveiling of the dependence of the exponent ν on $F > 2$ might be worth the cost since such study would conclude the full characterization of the nonequilibrium phase transition of Axelrod's model in the square lattice.

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